Feynman-Kac formula for heat equation driven by fractional white noise

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Abstract

In this paper we establish a version of the Feynman-Kac formula for the stochastic heat equation with a multiplicative fractional Brownian sheet. We prove the smoothness of the density of the solution, and the Hölder regularity in the space and time variables.

1 Introduction

Consider the following heat equation on $\mathbb{R}^d$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \frac{1}{2}\Delta u + c(t,x)u \\ u(0,x) = f(x)\,, \end{cases} \tag{1.1}$$

where f is a bounded measurable function. If $c(t,x)$ is a continuous function of $(t,x) \in [0,\infty) \times \mathbb{R}^d$, then we have the well-known Feynman-Kac formula (see [4]) for the solution to above equation

$$u(t,x) = E\left[f(B_t^x) \exp\left(\int_0^t c(t-s, B_s^x) ds \right) \right],$$

where $B_t^x = B_t + x$ is a d-dimensional Brownian motion starting from the point x.

In this paper, we shall extend the above Feynman-Kac formula to the heat equation with fractional noise

$$\begin{cases} \dfrac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \dfrac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d} \\ u(0,x) = f(x)\,, \end{cases} \tag{1.2}$$

*D. Nualart is supported by the NSF grant DMS0604207.

where $W(t,x)$ is a fractional Brownian sheet with Hurst parameters H_0 in time and $(H_1,\dots,H_d)$ in space, respectively. The difference between (1.1) and (1.2) is that $\frac{\partial^{d+1}W}{\partial t\partial x_1\cdots\partial x_d}$ is no longer a function of t and x but a generalized (random) function. For this equation, we can still formally write down the Feynman-Kac formula

$$u(t,x)=E^B\left[f(B_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta(B_{t-r}^x-y)W(dr,dy)\right)\right], \tag{1.3}$$

where E^B denotes the expectation with respect to the Brownian motion B_t^x, and δ denotes the Dirac delta function.

The aim of this paper is to justify the above formula (1.3), to show that the process $u(t,x)$ is a weak solution to Equation (1.2) and to establish some properties of this process. First, we shall show that $V_{t,x}:=\int_0^t\int_{\mathbb{R}^d}\delta(B_{t-r}^x-y)W(dr,dy)$ is a well-defined random variable. This will be done in Section 2 using a suitable approximation of the Dirac delta function, assuming that the Hurst parameters satisfy $2H_0+\sum_{i=1}^d H_i>d+1$, $H_0\ge\frac{1}{2}$, and $H_i>\frac{1}{2}$ for $1\le i\le d$.

After the definition of the random variable $V_{t,x}$, the next problem is to show its exponential integrability. With the use of the covariance structure of the fractional Brownian sheet $W(t,x)$, we show that $u(t,x)$ has exponential moments provided

$$E\exp\left[\lambda\int_0^1\int_0^1|r-s|^{2H_0-2}\prod_{i=1}^d|B_r^i-B_s^i|^{2H_i-2}drds\right]<\infty, \tag{1.4}$$

for any $\lambda\in\mathbb{R}$. To show that (1.4) is true we use a method of Le Gall ([9]) together with several other techniques. This is done in Section 3.

Another main point of this paper is to show that $u(t,x)$ defined by (1.3) is the solution to (1.2). Instead of following the classical approach based on Itô's formula, which seems complicated in our situation, we use again the approximation technique together with Malliavin calculus. This is the contents of Section 4.

In the above equation (1.2) the solution and the noise are multiplied using the ordinary product. This gives rise to the Stratonovitch integral when we interpret the equation in its integral form. However, there is a number of papers where Wick product between the solution and the noise is used, which corresponds to the Skorohod integral. We shall obtain the Feynman-Kac formula for Skorohod type equation in Section 8.

Feynman-Kac formula gives an explicit form of the solution. This explicit form has several consequences. First, in Section 5, we study the smoothness of the density of the probability law of $u(t,x)$ (with respect to the Lebesgue measure). In Section 6, we use the Feynman-Kac formula to obtain some Hölder continuity properties of the solution $u(t,x)$ with respect to t and x.

The above techniques work for $H_i>1/2$, $i=1,2,\dots,d$. From the condition $2H_0+\sum_{i=1}^d H_i>d+1$ it follows that H_0 must be greater than $1/2$ and we cannot

allow more than one of the $H_1, \ldots, H_d$ to be less than or equal to 1/2. Thus if we want to remove the condition $H_i > 1/2$, $i = 1, 2, \ldots, d$, we need $d = 1$. We show in Section 7 that if $H_1 = \frac{1}{2}$ and $H_0 > \frac{3}{4}$ then all previous results hold except the smoothness of the density. When $d = 1$, we can also handle the case $H_0 < 1/2$, assuming that the process has a regular spacial covariance. This has been done in the companion paper [5] using different techniques. We would like to mention that this type of Feynman-Kac formula was mentioned as a conjecture in the paper [10].

The appendix contains some technical results used along the paper.

2 Preliminaries

Fix a vector of Hurst parameters $H = (H_0, H_1, \ldots, H_d)$, where $H_i \in \left(\frac{1}{2}, 1\right)$. Suppose that $W = \{W(t, x), t \geq 0, x \in \mathbb{R}^d\}$ is a zero mean Gaussian random field with the covariance function

$$E(W(t,x)W(s,y)) = R_{H_0}(s,t) \prod_{i=1}^{d} R_{H_i}(x_i, y_i),$$

where for any $H \in (0,1)$ we denote by $R_H(s,t)$, the covariance function of the fractional Brownian motion with Hurst parameter H, that is,

$$R_H(s,t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

In other words, W is a fractional Brownian sheet with Hurst parameters H_0 in time variable and H_i in space variables, $i = 1, \ldots, d$.

Denote by $\mathcal{E}$ the linear span of the indicator functions of rectangles of the form $(s,t] \times (x,y]$ in $\mathbb{R}_+ \times \mathbb{R}^d$. Consider in $\mathcal{E}$ the inner product defined by

$$\langle I_{(0,s]\times(0,x]}, I_{(0,t]\times(0,y]} \rangle_{\mathcal{H}} = R_{H_0}(s,t) \prod_{i=1}^{d} R_{H_i}(x_i, y_i).$$

In the above formula, if $x_i < 0$ we assume by convention that $I_{(0,x_i]} = -I_{(-x_i,0]}$. We denote by $\mathcal{H}$ the closure of $\mathcal{E}$ with respect to this inner product. The mapping $W : I_{(0,t]\times(0,x]} \to W(t,x)$ extends to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by W. We will denote this isometry by

$$W(\phi) = \int_0^\infty \int_{\mathbb{R}^d} \phi(t,x) W(dt, dx),$$

if $\phi \in \mathcal{H}$. Notice that if ϕ and ψ are functions in $\mathcal{E}$, then

$$\begin{aligned} E\left(W(\phi)W(\psi)\right) &= \langle \phi, \psi \rangle_{\mathcal{H}} = \alpha_H \\ &\times \int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} \phi(s,x)\psi(t,y)|s-t|^{2H_0-2} \prod_{i=1}^{d} |x_i - y_i|^{2H_i-2} ds dt dx dy, \end{aligned}$$

where $\alpha_H = \prod_{i=0}^d H_i(2H_i - 1)$. Furthermore, $\mathcal{H}$ contains the class of measurable functions ϕ on $\mathbb{R}_+ \times \mathbb{R}^d$ such that

$$\int_{\mathbb{R}_+^2 \times \mathbb{R}^{2d}} |\phi(s,x)\phi(t,y)||s-t|^{2H_0-2} \prod_{i=1}^d |x_i - y_i|^{2H_i-2} dsdtdxdy < \infty.$$

We will denote by D the derivative operator in the sense of Malliavin calculus. That is, if F is a smooth and cylindrical random variable of the form

$$F = f(W(\phi_1), \dots, W(\phi_n)),$$

$\phi_i \in \mathcal{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth), then DF is the $\mathcal{H}$-valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n))\phi_j.$$

The operator D is closable form $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{E(F^2) + E(\|DF\|_{\mathcal{H}}^2)}.$$

We denote by δ the adjoint of the derivative operator, given by duality formula

$$E(\delta(u)F) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \tag{2.1}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \mathcal{H})$ in the domain of δ. The operator δ is also called the Skorohod integral because in the case of the Brownian motion it coincides with an extension of the Itô integral introduced by Skorohod. We refer to Nualart [11] for a detailed account on the Malliavin calculus with respect to a Gaussian process. We recall the following formula, which will be used in the paper

$$FW(\phi) = \delta(F\phi) + \langle DF, \phi\rangle_{\mathcal{H}}, \tag{2.2}$$

for any $\phi \in \mathcal{H}$ and any random variable F in the Sobolev space $\mathbb{D}^{1,2}$.

Along the paper C will denote a positive constant which may vary from one formula to another one.

3 Definition and exponential integrability of $\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y)W(dr, dy)$

For any $\varepsilon > 0$ we denote by $p_\varepsilon(x)$ the d-dimensional heat kernel

$$p_\epsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\epsilon}},\ x \in \mathbb{R}^d.$$

On the other hand, for any $\delta > 0$ we define the function

$$\varphi_\delta(x) = \frac{1}{\delta} I_{[0,\delta]}(x).$$

Then, $\varphi_\delta(t)p_\varepsilon(x)$ provides an approximation of the Dirac delta function $\delta(t,x)$ as ε and δ tend to zero. We denote by $W^{\epsilon,\delta}$ the approximation of the fractional Brownian sheet $W(t,x)$ defined by

$$W^{\epsilon,\delta}(t,x)=\int_0^t\int_{\mathbb{R}^d}\varphi_\delta(t-s)p_\epsilon(x-y)W(s,y)dsdy\,. \tag{3.1}$$

Fix $x\in\mathbb{R}^d$ and $t>0$. Suppose that $B=\{B_t,t\geq 0\}$ is a d-dimensional standard Brownian motion independent of W. We denote by $B_t^x=B_t+x$ the Brownian motion starting at the point x. We are going to define the random variable $\int_0^t\int_{\mathbb{R}^d}\delta(B_{t-r}^x-y)W(dr,dy)$ by approximating the Dirac delta function $\delta(B_{t-r}^x-y)$ by

$$A_{t,x}^{\epsilon,\delta}(r,y)=\int_0^t\varphi_\delta(t-s-r)p_\varepsilon(B_s^x-y)ds. \tag{3.2}$$

We will show that for any $\varepsilon>0$ and $\delta>0$ the function $A_{t,x}^{\epsilon,\delta}$ belongs to the space $\mathcal{H}$ almost surely, and the family of random variables

$$V_{t,x}^{\varepsilon,\delta}=\int_0^t\int_{\mathbb{R}^d}A_{t,x}^{\epsilon,\delta}(r,y)W(dr,dy)\,. \tag{3.3}$$

converges in L^2 as ε and δ tend to zero.

Along the paper we denote by $E^B(\Phi(B,W))$ (resp. by $E^W(\Phi(B,W))$) the expectation of a functional $\Phi(B,W)$ with respect to B (resp. with respect to W). We will use E for the composition E^BE^W, and also in case of a random variable depending only on B or W.

Theorem 3.1 *Suppose that* $2H_0+\sum_{i=1}^d H_i>d+1$. *Then, for any* $\varepsilon>0$ *and* $\delta>0$, $A_{t,x}^{\varepsilon,\delta}$ *defined in (3.2) belongs to* $\mathcal{H}$ *and the family of random variables* $V_{t,x}^{\epsilon,\delta}$ *defined in (3.3) converges in* L^2 *to a limit denoted by*

$$V_{t,x}=\int_0^t\int_{\mathbb{R}^d}\delta(B_{t-r}^x-y)W(dr,dy)\,. \tag{3.4}$$

Conditional to B, $V_{t,x}$ *is a Gaussian random variable with mean* 0 *and variance*

$$\mathrm{Var}^W(V_{t,x})=\alpha_H\int_0^t\int_0^t|r-s|^{2H_0-2}\prod_{i=1}^d\left|B_r^i-B_s^i\right|^{2H_i-2}drds\,. \tag{3.5}$$

Proof Fix ε, ε', δ and $\delta'>0$. Let us compute the inner product

$$\begin{aligned}\left\langle A_{t,x}^{\varepsilon,\delta},A_{t,x}^{\varepsilon',\delta'}\right\rangle_{\mathcal{H}} &= \alpha_H\int_{[0,t]^4}\int_{\mathbb{R}^{2d}}p_\epsilon(B_s^x-y)p_{\epsilon'}(B_r^x-z)\\ &\quad\times\varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)\\ &\quad\times|u-v|^{2H_0-2}\prod_{i=1}^d|y_i-z_i|^{2H_i-2}dydzdudvdsdr.\end{aligned} \tag{3.6}$$

By lemmas 9.2 and 9.3 we have the estimate

$$\begin{aligned}
&\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} p_\epsilon(B_s^x-y)p_{\epsilon'}(B_r^x-z)\\
&\times\varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)|u-v|^{2H_0-2}\prod_{i=1}^d|y_i-z_i|^{2H_i-2}dydzdudv\\
\leq\ & C|s-r|^{2H_0-2}\prod_{i=1}^d\left|B_s^i-B_r^i\right|^{2H_i-2}, \qquad (3.7)
\end{aligned}$$

for some constant $C>0$. The expectation of this random variable is integrable in $[0,t]^2$ because

$$\begin{aligned}
&E^B\int_0^t\int_0^t|s-r|^{2H_0-2}\prod_{i=1}^d\left|B_s^i-B_r^i\right|^{2H_i-2}dsdr\\
=\ &\prod_{i=1}^d E|\xi|^{2H_i-2}\int_0^t\int_0^t|s-r|^{2H_0+\sum_{i=1}^d H_i-d-2}dsdr\\
=\ &\frac{2\prod_{i=1}^d E|\xi|^{2H_i-2}t^{\kappa+1}}{\kappa(\kappa+1)}<\infty, \qquad (3.8)
\end{aligned}$$

where

$$\kappa=2H_0+\sum_{i=1}^d H_i-d-1>0. \qquad (3.9)$$

and ξ is a $N(0,1)$ random variable. The above computations show that the condition $2H_0+\sum_{i=1}^d H_i>d+1$ is necessary for the random variable $V_{t,x}$ to be well defined.

As a consequence, taking the mathematical expectation with respect to B in Equation (3.6), letting $\varepsilon=\varepsilon'$ and $\delta=\delta'$ and using the estimates (3.7) and (3.8) yields

$$E^B\left\|A_{t,x}^{\varepsilon,\delta}\right\|_{\mathcal{H}}^2\leq C.$$

This implies that almost surely $A_{t,x}^{\varepsilon,\delta}$ belongs to the space $\mathcal{H}$ for all ε and $\delta>0$. Therefore, the random variables $V_{t,x}^{\varepsilon,\delta}=W(A_{t,x}^{\varepsilon,\delta})$ are well defined and we have

$$E^BE^W(V_{t,x}^{\epsilon,\delta}V_{t,x}^{\epsilon',\delta'})=E^B\left\langle A_{t,x}^{\varepsilon,\delta},A_{t,x}^{\varepsilon',\delta'}\right\rangle_{\mathcal{H}}.$$

For any $s\neq r$ and $B_s\neq B_r$, as ε, ε', δ and δ' tend to zero, the left-hand side of the inequality (3.7) converges to $|s-r|^{2H_0-2}\prod_{i=1}^d\left|B_s^i-B_r^i\right|^{2H_i-2}$. Therefore, by dominated convergence theorem we obtain that $E^BE^W(V_{t,x}^{\epsilon,\delta}V_{t,x}^{\epsilon',\delta'})$ converges to Σ_t as ε, ε', δ and δ' tend to zero, where

$$\Sigma_t=\frac{2\alpha_H\prod_{i=1}^d E|\xi|^{2H_i-2}t^{\kappa+1}}{\kappa(\kappa+1)}.$$

This implies that $V_{t,x}^{\epsilon,\delta}$ converges in L^2 as ε and δ tend to zero to a limit denoted by $V_{t,x}$. Finally, by a similar argument we show (3.5). ■

The next result provides the exponential integrability of the random variable $V_{t,x}$ defined in (3.4).

Theorem 3.2 *Suppose that $2H_0 + \sum_{i=1}^d H_i > d+1$. Then, for any $\lambda \in \mathbb{R}$, we have*

$$E \exp\left(\lambda \int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy)\right) < \infty\,. \tag{3.10}$$

Proof The proof will be done in several steps.

Step 1 From (3.5) we obtain

$$Ee^{\lambda V_{t,x}} = E^B \exp\left(\frac{\lambda^2}{2}\alpha_H \int_0^t \int_0^t |s-r|^{2H_0-2} \prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2} dsdr\right),$$

and the scaling property of the Brownian motion yields

$$Ee^{\lambda V_{t,x}} = Ee^{\mu Y}, \tag{3.11}$$

where $\mu = \frac{\lambda^2}{2}\alpha_H t^{\kappa+1}$, where κ has been defined in (3.9), and

$$Y = \int_0^1 \int_0^1 |s-r|^{2H_0-2} \prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2} dsdr. \tag{3.12}$$

Then, it suffices to show that the random variable Y has exponential moments of all orders.

Step 2 Our approach to prove that $E\exp(\lambda Y) < \infty$ for any $\lambda \in \mathbb{R}$ is motivated by the method of Le Gall [9]. For $k = 1, \dots, 2^{n-1}$ we denote $A_{n,k} = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right] \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right]$ and define

$$\alpha_{n,k} = \int_{A_{n,k}} |s-r|^{2H_0-2} \prod_{i=1}^d |B_s^i - B_r^i|^{2H_i-2} dsdr.$$

The random variables $\alpha_{n,k}$ have the following two properties:

(i) For every $n \geq 1$, the variables $\alpha_{n,1}, \dots, \alpha_{n,2^{n-1}}$ are independent.

(ii) $\alpha_{n,k} \overset{d}{=} 2^{-n(\kappa+1)}\alpha_0$, where

$$\alpha_0 = \int_0^1 \int_0^1 (s+r)^{2H_0-2} \prod_{i=1}^d |B_s^i - \tilde{B}_r^i|^{2H_i-2} dsdr,$$

and $\tilde{B}$ is a standard Brownian motion independent of B.

The condition $2H_0 + \sum_{i=1}^d H_i > d+1$ implies that $E\alpha_0 < \infty$ and we deduce that

$$Y = 2\sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} \alpha_{n,k},$$

where the series converges in the L^1 sense.

Step 3 For any integer $n \geq 1$, we claim that

$$E\alpha_0^n \leq E\left(C\int_0^1 \prod_{i=1}^d |B_s^i|^{2H_i-2} ds\right)^n, \tag{3.13}$$

for some constant $C > 0$. In fact, we have

$$E\alpha_0^n = E\int_{[0,1]^{2n}} \prod_{j=1}^n \prod_{i=1}^d (|s_j + t_j|^{2H_0-2} |B_{s_j}^i - \tilde{B}_{t_j}^i|^{2H_i-2}) ds dt. \tag{3.14}$$

Using the formula

$$c^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty e^{-c\tau}\tau^{z-1} d\tau,$$

we obtain for each $i = 1, \dots, d$,

$$\begin{aligned} E\prod_{j=1}^n |B_{s_j}^i - \tilde{B}_{t_j}^i|^{2H_i-2} &= \Gamma(1-H_i)^{-n} \\ &\times \int_{[0,\infty)^n} E\exp\left(-\sum_{j=1}^n |B_{s_j}^i - \tilde{B}_{t_j}^i|^2 \tau_j\right) \prod_{j=1}^n \tau_j^{-H_i} d\tau. \end{aligned} \tag{3.15}$$

For any $\tau_1, \dots, \tau_n > 0$ and $s_1, t_1, \dots, s_n, t_n \in (0,1)$, we denote

$$Q_1 = \left(E(B_{s_j}^i B_{s_k}^i)\sqrt{\tau_j \tau_k}\right)_{n\times n}, \quad Q_2 = \left(E(\tilde{B}_{t_j}^i \tilde{B}_{t_k}^i)\sqrt{\tau_j\tau_k}\right)_{n\times n}.$$

We know that

$$E\exp\left(-\sum_{j=1}^n |B_{s_j}^i - \tilde{B}_{t_j}^i|^2 \tau_j\right) = \det(I + 2Q_1 + 2Q_2)^{-\frac{1}{2}}. \tag{3.16}$$

Substituting (3.16) into (3.15) yields

$$
\begin{aligned}
& E\prod_{j=1}^{n}|B^i_{s_j}-\tilde{B}^i_{t_j}|^{2H_i-2}=\Gamma(1-H_i)^{-n} \\
& \times\int_{[0,\infty)^n}\det(I+2Q_1+2Q_2)^{-\frac{1}{2}}\prod_{j=1}^{n}\tau_j^{-H_i}d\tau \\
\leq\ & \Gamma(1-H_i)^{-n}\int_{[0,\infty)^n}\det(I+2Q_1)^{-\frac{1}{4}}\det(I+2Q_2)^{-\frac{1}{4}}\prod_{j=1}^{n}\tau_j^{-H_i}d\tau \\
\leq\ & \Gamma(1-H_i)^{-n}\ \left[\int_{[0,\infty)^n}\det(I+2Q_1)^{-\frac{1}{2}}\prod_{j=1}^{n}\tau_j^{-H_i}d\tau\right]^{\frac{1}{2}} \\
& \times\left[\int_{[0,\infty)^n}\det(I+2Q_2)^{-\frac{1}{2}}\prod_{j=1}^{n}\tau_j^{-H_i}d\tau\right]^{\frac{1}{2}} \\
=\ & \left[E\prod_{j=1}^{n}|B^i_{s_j}|^{2H_i-2}E\prod_{j=1}^{n}|\tilde{B}^i_{t_j}|^{2H_i-2}\right]^{\frac{1}{2}}, \qquad (3.17)
\end{aligned}
$$

where in the above first inequality, we have used the estimates

$$
(I+2Q_1+2Q_2)\geq\frac{1}{2}[(I+2Q_1)+(I+2Q_2)]\geq(I+2Q_1)^{\frac{1}{2}}(I+2Q_2)^{\frac{1}{2}}.
$$

Substituting (3.17) into (3.14), and using the inequality $(s_j+t_j)^{2H_0-2}\leq s_j^{H_0-1}t_j^{H_0-1}$, we obtain

$$
\begin{aligned}
E\alpha_0^n\ &\leq\ \int_{[0,1]^{2n}}\prod_{j=1}^{n}(s_j+t_j)^{2H_0-2}\prod_{i=1}^{d}\left[E\prod_{j=1}^{n}|B^i_{s_j}|^{2H_i-2}E\prod_{j=1}^{n}|\tilde{B}^i_{t_j}|^{2H_i-2}\right]^{\frac{1}{2}}dsdt \\
&\leq\ \left(\int_{[0,1]^n}\prod_{j=1}^{n}s_j^{H_0-1}\left(E\prod_{j=1}^{n}\prod_{i=1}^{d}|B^i_{s_j}|^{2H_i-2}\right)^{\frac{1}{2}}ds\right)^2.
\end{aligned}
$$

Finally, using Hölder's inequality with $\frac{1}{H_0}<p<2$ we get

$$
\begin{aligned}
E\alpha_0^n\ &\leq\ C^n\left(\int_{[0,1]^n}\left(E\prod_{i=1}^{d}\prod_{j=1}^{n}|B^i_{s_j}|^{2H_i-2}\right)^{\frac{p}{2}}ds\right)^{\frac{2}{p}} \\
&\leq\ C^n\int_{[0,1]^n}E\prod_{i=1}^{d}\prod_{j=1}^{n}|B^i_{s_j}|^{2H_i-2}ds \\
&=\ E\left(C\int_0^1\prod_{i=1}^{d}|B^i_s|^{2H_i-2}ds\right)^n.
\end{aligned}
$$

This completes the proof of (3.13).

Step 4 For any $\lambda > 0$, using (3.13) and Lemma 9.5 in the Appendix we obtain

$$Ee^{\lambda\alpha_0} \le E\exp\left(C\lambda\int_0^1 \prod_{i=1}^d |B_s^i|^{2H_i-2}ds\right) < \infty, \tag{3.18}$$

because $\rho < 1$.

Step 5 Define $\varphi(\lambda) = E(e^{\lambda(\alpha_0 - E\alpha_0)})$. By (3.18), $\varphi(\lambda) < \infty$ for all $\lambda \in \mathbb{R}$. Since $\varphi'(0) = 0$, for every $K > 0$ we can find a positive constant C_K such that for all $\lambda \in [0, K]$

$$\varphi(\lambda) \le 1 + C_K\lambda^2.$$

Define $\overline{\alpha}_{n,k} = \alpha_{n,k} - E(\alpha_{n,k})$. Fix $K > 0$ and $a \in (0, \kappa+1)$, where κ has been introduced in (3.9). Recall that by property (ii) in Step 3, $\overline{\alpha}_{n,k} \overset{d}{=} 2^{-n(\kappa+1)}\overline{\alpha}_0$. For every $N \ge 2$ set $b_N = 2K\prod_{j=2}^{j=N}(1-2^{-a(j-1)})$ and set $b_1 = 2K$. Then by Hölder's inequality and properties (i) and (ii) of $\alpha_{n,k}$, we have for $N \ge 2$,

$$\begin{aligned}
&E\exp\left(b_N\sum_{n=1}^N\sum_{k=1}^{2^{n-1}}\overline{\alpha}_{n,k}\right)\\
&\le \left[E\exp\left(\frac{b_N}{1-2^{-a(N-1)}}\sum_{n=1}^{N-1}\sum_{k=1}^{2^{n-1}}\overline{\alpha}_{n,k}\right)\right]^{1-2^{-a(N-1)}}\\
&\quad\times\left[E\exp\left(2^{a(N-1)}b_N\sum_{k=1}^{2^{N-1}}\overline{\alpha}_{N,k}\right)\right]^{2^{-a(N-1)}}\\
&\le E\exp\left(b_{N-1}\sum_{n=1}^{N-1}\sum_{k=1}^{2^{n-1}}\overline{\alpha}_{n,k}\right)\varphi(b_N2^{a(N-1)-(\kappa+1)N})^{2^{(1-a)(N-1)}}.
\end{aligned}$$

Notice that $b_N2^{a(N-1)-(\kappa+1)N} \le 2K$. It follows that,

$$\begin{aligned}
\varphi(b_N2^{a(N-1)-(\kappa+1)N})^{2^{(1-a)(N-1)}} &\le \left(1 + C_Kb_N^2 2^{2((a-\kappa-1)N-a)}\right)^{2^{(1-a)(N-1)}}\\
&\le \exp(C2^{(a+1-2(\kappa+1))N})
\end{aligned}$$

for a constant C independent of N. By induction we get

$$\begin{aligned}
E\exp\left(b_N\sum_{n=1}^N\sum_{k=1}^{2^{n-1}}\overline{\alpha}_{n,k}\right) &\le \exp\left(C\sum_{n=2}^N 2^{(a+1-2(\kappa+1))n}\right)E\exp\left(b_1\overline{\alpha}_{1,1}\right)\\
&\le \exp\left(C(1-2^{a+1-2(\kappa+1)})^{-1}\right)\varphi(K).
\end{aligned}$$

Letting N tend to infinity and using Fatou's lemma, we obtain

$$E\exp\left(b_\infty(Y-EY)/2\right)<\infty,$$

where $b_\infty = 2K\prod_{j=1}^\infty(1-2^{-aj})>0$. Since $K>0$ is arbitrary, we conclude that $E\exp(\lambda Y)<\infty$ for all $\lambda\in\mathbb{R}$. This completes the proof, in view of (3.11). ■

4 Feynman-Kac formula

We recall that W is a fractional Brownian sheet on $\mathbb{R}_+\times\mathbb{R}^d$ with Hurst parameters $(H_0,H_1,\ldots,H_d)$ where $H_i\in(\frac{1}{2},1)$ for $i=0,\ldots,d$. For any ε, $\delta>0$ we define

$$\dot W^{\epsilon,\delta}(t,x):=\int_0^t\int_{\mathbb{R}^d}\varphi_\delta(t-s)p_\epsilon(x-y)W(ds,dy).$$

In order to give a notion of solution for the heat equation with fractional noise (1.2) we need the following definition of the Stratonovitch integral.

Definition 4.1 *Given a random field $v=\{v(t,x),t\ge0,x\in\mathbb{R}^d\}$ such that*

$$\int_0^T\int_{\mathbb{R}^d}|v(t,x)|dxdt<\infty$$

almost surely for all $T>0$, the Stratonovitch integral $\int_0^T\int_{\mathbb{R}^d}v(t,x)W(dt,dx)$ is defined as the following limit in probability if it exists

$$\lim_{\epsilon,\delta\downarrow0}\int_0^T\int_{\mathbb{R}^d}v(t,x)\dot W^{\epsilon,\delta}(t,x)dxdt.$$

We are going to consider the following notion of solution for Equation (1.2).

Definition 4.2 *A random field $u=\{u(t,x),t\ge0,x\in\mathbb{R}^d\}$ is a weak solution to Equation (1.2) if for any C^∞ function φ with compact support on $\mathbb{R}^d$, we have*

$$\begin{aligned}\int_{\mathbb{R}^d}u(t,x)\varphi(x)dx &= \int_{\mathbb{R}^d}f(x)\varphi(x)dx+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}u(s,x)\Delta\varphi(x)dxds\\ &\quad+\int_0^t\int_{\mathbb{R}^d}u(s,x)\varphi(x)W(ds,dx),\end{aligned}$$

almost surely, for all $t\ge0$, where the last term is a Stratonovitch stochastic integral in the sense of Definition 4.1.

The following is the main result of this section.

Theorem 4.3 *Suppose that $2H_0+\sum_{i=1}^d H_i>d+1$ and that f is a bounded measurable function. Then the process*

$$u(t,x)=E^B\left(f(B_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta(B_{t-r}^x-y)W(dr,dy)\right)\right)\qquad(4.1)$$

is a weak solution to Equation (1.2).

Proof Consider the approximation of the Equation (1.2) given by the following heat equation with a random potential

$$\begin{cases} \dfrac{\partial u^{\varepsilon,\delta}}{\partial t} = \frac{1}{2}\Delta u^{\varepsilon,\delta} + u^{\varepsilon,\delta}\dot{W}^{\epsilon,\delta}_{t,x} \\ u^{\varepsilon,\delta}(0,x) = f(x). \end{cases} \tag{4.2}$$

From the classical Feynman-Kac formula we know that

$$u^{\varepsilon,\delta}(t,x) = E^B\left(f(B^x_t)\exp\left(\int_0^t \dot{W}^{\epsilon,\delta}(t-s,B^x_s)ds\right)\right),$$

where B^x_t is a d-dimensional Brownian motion independent of W starting at x. By Fubini's theorem we can write

$$\begin{aligned} \int_0^t \dot{W}^{\varepsilon,\delta}(t-s,B^x_s)ds &= \int_0^t \left(\int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s-r)p_\varepsilon(B^x_s-y)W(dr,dy)\right)ds \\ &= \int_0^t \int_{\mathbb{R}^d}\left(\int_0^t \varphi_\delta(t-s-r)p_\varepsilon(B^x_s-y)ds\right)W(dr,dy) \\ &= V^{\varepsilon,\delta}_{t,x}, \end{aligned}$$

where $V^{\varepsilon,\delta}_{t,x}$ is defined in (3.3). Therefore,

$$u^{\varepsilon,\delta}(t,x) = E^B\left(f(B^x_t)\exp\left(V^{\varepsilon,\delta}_{t,x}\right)\right).$$

Step 1 We will prove that for any $x \in \mathbb{R}^d$ and any $t > 0$, we have

$$\lim_{\varepsilon,\delta\downarrow 0} E^W|u^{\varepsilon,\delta}(t,x) - u(t,x)|^p = 0, \tag{4.3}$$

for all $p \geq 2$, where $u(t,x)$ is defined in (4.1). Notice that

$$\begin{aligned} E^W|u^{\varepsilon,\delta}(t,x) - u(t,x)|^p &= E^W\left|E^B\left(f(B^x_t)\left[\exp\left(V^{\varepsilon,\delta}_{t,x}\right) - \exp\left(V_{t,x}\right)\right]\right)\right|^p \\ &\leq \|f\|^p_\infty\, E\left|\exp\left(V^{\varepsilon,\delta}_{t,x}\right) - \exp\left(V_{t,x}\right)\right|^p, \end{aligned}$$

where $V_{t,x}$ is defined in (3.4). Since $\exp\left(V^{\varepsilon,\delta}_{t,x}\right)$ converges to $\exp\left(V_{t,x}\right)$ in probability by Theorem 3.1, to show (4.3) it suffices to prove that for any $\lambda \in \mathbb{R}$

$$\sup_{\epsilon,\delta} E\exp\left(\lambda V^{\varepsilon,\delta}_{t,x}\right) < \infty. \tag{4.4}$$

The estimate (4.4) follows from (3.3), (3.7), and (3.10):

$$\begin{aligned} E\exp\left(\lambda V^{\varepsilon,\delta}_{t,x}\right) &= E\exp\left(\frac{\lambda^2}{2}\left\|A^{\varepsilon,\delta}_{t,x}\right\|^2_{\mathcal{H}}\right) \\ &\leq E\exp\left(\frac{\lambda^2}{2}C\int_0^t\int_0^t |r-s|^{2H_0-2}\prod_{i=1}^d \left|B^i_r - B^i_s\right|^{2H_i-2} drds\right) \\ &< \infty. \end{aligned} \tag{4.5}$$

Step 2 Now we prove that $u(t,x)$ is a weak solution to Equation (1.2) in the sense of Definition 4.2. Suppose φ is a smooth function with compact support. We know that,

$$\begin{aligned}\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\varphi(x)dx &= \int_{\mathbb{R}^d} f(x)\varphi(x)dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\Delta\varphi(x)dxds \\ &\quad + \int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(t,x)\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)dsdx. \qquad (4.6)\end{aligned}$$

Therefore, it suffices to prove that

$$\lim_{\varepsilon,\delta\downarrow 0}\int_0^t\int_{\mathbb{R}^d} u^{\varepsilon,\delta}(s,x)\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)dsdx = \int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds,dx),$$

in probability. From (4.6) and (4.3) it follows that $\int_0^t\int_{\mathbb{R}^d} u^{\epsilon,\delta}(s,x)\varphi(x)\dot{W}^{\epsilon,\delta}(s,x)dsdx$ converges in L^2 to the random variable

$$G = \int_{\mathbb{R}^d} u(t,x)\varphi(x)dx - \int_{\mathbb{R}^d} f(x)\varphi(x)dx - \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u(t,x)\Delta\varphi(x)dxds$$

as ε and δ tend to zero. Hence, if

$$B_{\varepsilon,\delta} = \int_0^t\int_{\mathbb{R}^d}(u^{\epsilon,\delta}(s,x)-u(s,x))\varphi(x)\dot{W}^{\varepsilon,\delta}(s,x)dsdx$$

converges in L^2 to zero, $u(s,x)\varphi(x)$ will be Stratonovitch integrable and

$$\int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds,dx) = G\,,$$

which completes the proof. In order to show the convergence to zero of $B_{\varepsilon,\delta}$, we will express the product $(u^{\varepsilon,\delta}(s,x)-u(s,x))\dot{W}^{\varepsilon,\delta}(s,x)$ as the sum of a divergence integral plus a trace term (see (2.2))

$$\begin{aligned}&(u^{\varepsilon,\delta}(s,x)-u(s,x))\dot{W}^{\varepsilon,\delta}(s,x) \\ &\quad = \int_0^t\int_{\mathbb{R}^d}(u^{\varepsilon,\delta}(s,x)-u(s,x))\varphi_\delta(s-r)p_\varepsilon(x-z)\delta W_{r,z} \\ &\qquad + \langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)),\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}}\,.\end{aligned}$$

Then we have

$$\begin{aligned}B_{\varepsilon,\delta} &= \int_0^t\int_{\mathbb{R}^d}\phi^{\epsilon,\delta}_{r,z}\delta W_{r,z} \\ &\quad + \int_0^t\int_{\mathbb{R}^d}\varphi(x)\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)),\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}}dsdx \\ &= B^1_{\varepsilon,\delta}+B^2_{\varepsilon,\delta}, \qquad (4.7)\end{aligned}$$

where

$$\phi^{\varepsilon,\delta}_{r,z} = \int_0^t \int_{\mathbb{R}^d} (u^{\varepsilon,\delta}(s,x) - u(s,x))\varphi(x)\varphi_\delta(s-r)p_\varepsilon(x-z)dsdx,$$

and $\delta(\phi^{\varepsilon,\delta}) = \int_0^t \int_{\mathbb{R}^d} \phi^{\varepsilon,\delta}_{r,z}\delta W_{r,z}$ denotes the divergence or the Skorohod integral of $\phi^{\varepsilon,\delta}$. For the term $B^1_{\varepsilon,\delta}$ we use the following L^2 estimate for the Skorohod integral

$$E[(B^1_{\varepsilon,\delta})^2] \leq E(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}}) + E(\|D\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}\otimes\mathcal{H}})\,. \tag{4.8}$$

The first term in (4.8) is estimated as follows

$$\begin{aligned} E(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}}) &= \int_0^t \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} E\left[(u^{\varepsilon,\delta}(s,x) - u(s,x))(u^{\varepsilon,\delta}(r,y) - u(r,y))\right] \\ &\quad \times\varphi(x)\varphi(y)\langle\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot), \varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}}dsdxdrdy. \end{aligned} \tag{4.9}$$

Using lemmas 9.2 and 9.3 we can write

$$\begin{aligned} &\langle\varphi_\delta(s-\cdot)p_\epsilon(x-\cdot), \varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}} \\ =\; & \alpha_H\left(\int_{[0,t]^2} \varphi_\delta(s-\sigma)\varphi_\delta(r-\tau)|\sigma-\tau|^{2H_0-2}d\sigma d\tau\right) \\ & \times\left(\int_{\mathbb{R}^{2d}} p_\varepsilon(x-z)p_\varepsilon(y-w)\prod_{i=1}^d |z_i-w_i|^{2H_i-2}dzdw\right) \\ \leq\; & C|s-r|^{2H_0-2}\prod_{i=1}^d |x-y|^{2H_i-2}, \end{aligned} \tag{4.10}$$

for some constant $C>0$. As a consequence, the integrand on the right-hand side of Equation (4.9) converges to zero as ε and δ tend to zero for any s, r, x, y due to (4.3). From (4.5) we get

$$\sup_{\varepsilon,\delta}\sup_{x\in\mathbb{R}^d}\sup_{0\leq s\leq t} E\left(u^{\epsilon,\delta}(s,x)\right)^2 \leq \|f\|^2_\infty \sup_{\varepsilon,\delta}\sup_{x\in\mathbb{R}^d}\sup_{0\leq s\leq t} E\exp\left(2V^{\varepsilon,\delta}_{s,x}\right) < \infty. \tag{4.11}$$

Hence, from (4.10) and (4.11) we get that the integrand on the right-hand side of Equation (4.9) is bounded by $C|s-r|^{2H_0-2}\prod_{i=1}^d |x_i-y_i|^{2H_i-2}$, for some constant $C>0$. Therefore, by dominated convergence we get that $E(\|\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}})$ converges to zero as ε and δ tend to zero.

On the other hand, we have

$$D(u^{\varepsilon,\delta}(t,x)) = E^B\left[f(B_t+x)\exp(V^{\varepsilon,\delta}_{t,x})A^{\varepsilon,\delta}_{t,x}\right],$$

where $A^{\varepsilon,\delta}_{t,x}$ is defined in (3.2). Therefore,

$$\begin{aligned} &E\langle D(u^{\varepsilon,\delta}(t,x)), D(u^{\varepsilon',\delta'}(t,x))\rangle_{\mathcal{H}} \\ =\; & E^W E^B\Big(f(B^1_t+x)f(B^2_t+x) \\ & \times\exp(V^{\varepsilon,\delta}_{t,x}(B^1)+V^{\varepsilon,\delta}_{t,x}(B^2))\langle A^{\varepsilon,\delta}_{t,x}(B^1), A^{\varepsilon',\delta'}_{t,x}(B^2)\rangle_{\mathcal{H}}\Big), \end{aligned}$$

where B^1 and B^2 are two independent d-dimensional Brownian motions, and here E^B denotes the expectation with respect to (B^1, B^2). Then from the previous results it is easy to show that

$$\begin{aligned}
&\lim_{\varepsilon,\delta\downarrow 0} E\langle D(u^{\varepsilon,\delta}(t,x)), D(u^{\varepsilon',\delta'}(t,x))\rangle_{\mathcal{H}} \\
= \; & E\Big[f(B_t^1+x)f(B_t^2+x) \\
&\times \exp\left(\frac{\alpha_H}{2}\sum_{j,k=1}^{2}\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^{d}|B_s^{j,i}-B_r^{k,i}|^{2H_i-2}dsdr\right) \\
&\times \alpha_H\int_0^t\int_0^t |s-r|^{2H_0-2}\prod_{i=1}^{d}|B_s^{1,i}-B_r^{2,i}|^{2H_i-2}dsdr\Big]. \qquad (4.12)
\end{aligned}$$

This implies that $u^{\varepsilon,\delta}(t,x)$ converges in the space $\mathbb{D}^{1,2}$ to $u(t,x)$ as $\delta\downarrow 0$ and $\varepsilon\downarrow 0$. On the other hand, we also have

$$\sup_{\varepsilon,\delta}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le t} E\left\|D(u^{\varepsilon,\delta}(s,x))\right\|_{\mathcal{H}}^2<\infty.$$

Then

$$\begin{aligned}
E\|D\phi^{\varepsilon,\delta}\|^2_{\mathcal{H}\otimes\mathcal{H}} \; = \; & \int_0^t\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}} E\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)), D(u^{\varepsilon,\delta}(r,y)-u(r,y))\rangle_{\mathcal{H}} \\
&\times\varphi(x)\varphi(y)\langle\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot),\varphi_\delta(r-\cdot)p_\varepsilon(y-\cdot)\rangle_{\mathcal{H}}dsdxdrdy
\end{aligned}$$

converges to zero as ε and δ tend to zero. Hence, by (4.8) $B^1_{\varepsilon,\delta}$ converges to zero in L^2 as ε and δ tend to zero.

The second summand in the right-hand side of (4.7) can be written as

$$\begin{aligned}
B^2_{\varepsilon,\delta} \; = \; & \int_0^t\int_{\mathbb{R}^d}\varphi(x)\langle D(u^{\varepsilon,\delta}(s,x)-u(s,x)),\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}}dsdx \\
= \; & \int_0^t\int_{\mathbb{R}^d}\varphi(x)E^B\left(f(B_s^x)\exp\left(V_{s,x}^{\varepsilon,\delta}\right)\langle A_{s,x}^{\varepsilon,\delta},\varphi_\delta(s-\cdot)p_\epsilon(x-\cdot)\rangle_{\mathcal{H}}\right)dsdx \\
& -\int_0^t\int_{\mathbb{R}}\varphi(x)E^B\left(f(B_s^x)\exp\left(V_{s,x}\right)\langle\delta(B^x_{s-\cdot}-\cdot),\varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}}\right)dsdx \\
= \; & B^3_{\varepsilon,\delta}-B^4_{\varepsilon,\delta}
\end{aligned}$$

where

$$\begin{aligned}
\langle A_{s,x}^{\varepsilon,\delta},\varphi_\delta(s-\cdot)p_\epsilon(x-\cdot)\rangle_{\mathcal{H}} \; = \; & \alpha_H\int_{[0,s]^3}\int_{\mathbb{R}^{2d}}|r-v|^{2H_0-2}\prod_{i=1}^d|y_i-z_i|^{2H_i-2} \\
&\times\varphi_\delta(s-r)p_\varepsilon(B_r^x-y) \\
&\times\varphi_\delta(s-v)p_\varepsilon(x-z)dydyzdrdrdv,
\end{aligned}$$

and

$$
\begin{aligned}
&\langle \delta(B^x_{s-\cdot} - \cdot), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} \\
= \quad & \alpha_H \int_{[0,s]^2} \int_{\mathbb{R}^d} v^{2H_0-2} \prod_{i=1}^d |B^{x_i}_r - y_i|^{2H_i-2} \varphi_\delta(r-v) p_\varepsilon(x-y) dy dv dr.
\end{aligned}
$$

Lemma 9.2 and Lemma 9.3 imply that

$$
\langle A^{\varepsilon,\delta}_{s,x}, \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} \le C \int_0^s r^{2H_0-2} \prod_{i=1}^d |B^i_r|^{2H_i-2} dr, \tag{4.13}
$$

and

$$
\langle \delta(B^x_{s-\cdot} - \cdot), \varphi_\delta(s-\cdot)p_\varepsilon(x-\cdot)\rangle_{\mathcal{H}} \le C \int_0^s r^{2H_0-2} \prod_{i=1}^d |B^i_r|^{2H_i-2} dr, \tag{4.14}
$$

for some constant $C > 0$. Then, from (4.13) and (4.14) and from the fact that the random variable $\int_0^s r^{2H_0-2} \prod_{i=1}^d |B^i_r|^{2H_i-2} dr$ is square integrable because of Lemma 9.4, we can apply the dominated convergence theorem and get that $B^3_{\varepsilon,\delta}$ and $B^4_{\varepsilon,\delta}$ converge both in L^2 to

$$
\alpha_H \int_0^t \int_{\mathbb{R}^d} \varphi(x) E^B \left(f(B^x_s) \exp\left(V_{s,x}\right) \int_0^s r^{2H_0-2} \prod_{i=1}^d |B^i_r|^{2H_i-2} dr \right) ds dx,
$$

as ε and δ tend to zero. Therefore $B^2_{\varepsilon,\delta}$ converges in L^2 to zero as ε and δ tend to zero. This completes the proof. ■

Corollary 4.4 *Suppose $2H_0 + \sum_{i=1}^d H_i > d+1$. Then the solution $u(t,x)$ has finite moments of all orders. Moreover, for any positive integer p, we have*

$$
\begin{aligned}
E\left(u(t,x)^p\right) = E\Bigg(& \prod_{j=1}^p f(B^j_t + x) \\
& \times \exp\left[\frac{\alpha_H}{2} \sum_{j,k=1}^p \int_0^t \int_0^t |s-r|^{2H_0-2} \prod_{i=1}^d |B^{j,i}_s - B^{k,i}_r|^{2H_i-2} ds dr \right] \Bigg),
\end{aligned} \tag{4.15}
$$

where $B_1, \dots, B_p$ are independent d-dimensional standard Brownian motions.

Remark 4.5 *A formula similar to (4.15) for the solution to heat equation driven by fractional white noise for $H_1 = \cdots = H_d = \frac{1}{2}$ is discussed in [6].*

5 Regularity of the density

In this section we shall use the Feynman-Kac formula established in the previous section to show that for any t and x, the probability law of the solution $u(t,x)$ of Equation (1.2) has a smooth density with respect to the Lebesgue measure. To this end we shall show that $\|Du(t,x)\|_{\mathcal{H}}$ has negative moments of all orders.

Theorem 5.1 *Suppose that* $2H_0 + \sum_{i=1}^d H_i > d+1$. *Fix* $t > 0$ *and* $x \in \mathbb{R}^d$. *Assume that for any positive number* p, $E|f(B_t + x)|^{-p} < \infty$. *Then, the law of* $u(t,x)$ *has a smooth density.*

Proof From Theorem 4.3 we can write

$$u(t,x) = E^B \left[f(B_t^x) \exp\left(V_{t,x}\right) \right] .$$

The Malliavin derivative of the solution is given by

$$D_{r,y} u(t,x) = E^B \left[f(B_t^x) \exp\left(V_{t,x}\right) \delta(B_{t-r}^x - y) \right] .$$

It is not difficult to show that $u(t,x) \in \mathbb{D}^\infty$. Thus, by the general criterion for the smoothness of densities (see [11]), it suffices to show that $E\left(\|Du(t,x)\|_{\mathcal{H}}^{-2p} \right) < \infty$ for any $t > 0$ and $x \in \mathbb{R}^d$. We have

$$\begin{aligned}
\|Du(t,x)\|_{\mathcal{H}}^2 &= E^B \Big[f(B_t^1 + x) f(B_t^2 + x) \exp\left(V_{t,x}(B^1) + V_{t,x}(B^2)\right) \\
&\quad \times \langle \delta(B_{t-r}^{1,x} - y)\,, \delta(B_{t-r}^{2,x} - y) \rangle_{\mathcal{H}} \Big] \\
&= \alpha_H E^B \Big[f(B_t^1 + x) f(B_t^2 + x) \exp\left(V_{t,x}(B^1) + V_{t,x}(B^2)\right) \\
&\quad \times \int_0^t \int_0^t |r-s|^{2H_0 - 2} \prod_{i=1}^d |B_{t-r}^{1,i} - B_{t-s}^{2,i}|^{2H_i - 2} dr ds \Big] ,
\end{aligned}$$

where B^1 and B^2 are independent d-dimensional Brownian motions. By Jensen's inequality, we have for any $p > 0$,

$$\begin{aligned}
&\|Du(t,x)\|_{\mathcal{H}}^{-2p} \\
\le\ & (\alpha_H)^{-p} E^B \Big[|f(B_t^1 + x) f(B_t^2 + x)|^{-p} \exp\left(-p\left[V_{t,x}(B^1) + V_{t,x}(B^2)\right]\right) \\
&\times \left(\int_0^t \int_0^t |r-s|^{2H_0 - 2} \prod_{i=1}^d |B_{t-r}^{i,1} - B_{t-s}^{2,i}|^{2H_i - 2} dr ds \right)^{-p} \Big] .
\end{aligned}$$

Hence by Hölder's inequality, we obtain

$$\begin{aligned}
&E\, \|Du(t,x)\|_{\mathcal{H}}^{-2p} \\
\le\ & (\alpha_H)^{-p} \left(E|f(B_t^1 + x) f(B_t^2 + x)|^{-pp_1} \right)^{\frac{1}{p_1}} \\
&\times \left(E \exp\left(-pp_2 \left[V_{t,x}(B^1) + V_{t,x}(B^2)\right]\right) \right)^{\frac{1}{p_2}} \\
&\times \left(E \left(\int_0^t \int_0^t |r-s|^{2H_0 - 2} \prod_{i=1}^d |B_{t-r}^{1,i} - B_{t-s}^{2,i}|^{2H_i - 2} dr ds \right)^{-pp_3} \right)^{\frac{1}{p_3}} \\
=\ & I_1 I_2 I_3 \,,
\end{aligned}$$

where $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. The first factor I_1 is finite by the assumption on f and Hölder's inequality. The second factor is finite by Theorem 3.2. Finally, from

Jensen's inequality, we have

$$
\begin{aligned}
I_3^{p_3} &= E\left[t^{-2pp_3}\left\{\frac{1}{t^2}\int_0^t\int_0^t |r-s|^{2H_0-2}\prod_{i=1}^d |B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{2H_i-2}drds\right\}^{-pp_3}\right] \\
&\leq E\left[t^{-2pp_3-2}\left\{\int_0^t\int_0^t |r-s|^{-(2H_0-2)pp_3}\prod_{i=1}^d |B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{-(2H_i-2)pp_3}drds\right\}\right] \\
&\leq C\int_0^t\int_0^t |r-s|^{-(2H_0-2)pp_3} E\left\{\prod_{i=1}^d |B_{t-r}^{1,i}-B_{t-s}^{2,i}|^{-(2H_i-2)pp_3}\right\}drds \\
&< \infty\,.
\end{aligned}
$$

This completes the proof. ■

6 Hölder continuity of the solution

In this section, we study the Hölder continuity of the solution to the equation (1.2). The main result of this section is the following theorem.

Theorem 6.1 *Suppose that $2H_0+\sum_{i=1}^d H_i > d+1$ and let $u(t,x)$ be the solution of Equation (1.2). Then $u(t,x)$ has a continuous modification such that for any $\rho \in \left(0, \frac{\kappa}{2}\right)$ (where κ has been defined in (3.9)), and any compact rectangle $I \subset \mathbb{R}_+ \times \mathbb{R}^d$ there exists a positive random variable K_I such that almost surely, for any $(s,x),(t,y) \in I$ we have*

$$
|u(t,y)-u(s,x)| \leq K_I(|t-s|^\rho + |y-x|^{2\rho}).
$$

Proof The proof will be done in several steps.

Step 1 Recall that $V_{t,x} = \int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y)W(dr,dy)$ denotes the random variable introduced in (3.4) and

$$
u(t,x) = E^B\left(f(B_t^x)\exp\left(V(_{t,x})\right)\right).
$$

Set $V = V_{s,x}$ and $\tilde{V} = V_{t,y}$. Then we can write

$$
\begin{aligned}
&E^W|u(s,x)-u(t,y)|^p = E^W|E^B(e^V - e^{\tilde{V}})|^p \\
&\leq E^W\left(E^B[|\tilde{V}-V|e^{\max(V,\tilde{V})}]\right)^p \\
&\leq E^W\left[\left(E^B e^{2\max(V,\tilde{V})}\right)^{p/2}\left(E^B(\tilde{V}-V)^2\right)^{p/2}\right] \\
&\leq \left[E^W E^B e^{2p\max(V,\tilde{V})}\right]^{\frac{1}{2}}\left[E^W\left(E^B(\tilde{V}-V)^2\right)^p\right]^{\frac{1}{2}}.
\end{aligned}
$$

Applying Minkowski's inequality, the equivalence between the L^2 norm and the L^p norm for a Gaussian random variable, and using the exponential integrability property (3.10) we obtain

$$
\begin{aligned}
&E^W|u(s,x)-u(t,y)|^p \leq C\left[E^W\left(E^B(\tilde{V}-V)^2\right)^p\right]^{\frac{1}{2}} \\
&\leq C_p\left[E^B E^W|\tilde{V}-V|^2\right]^{p/2}. \qquad (6.1)
\end{aligned}
$$

In a similar way to (3.5) we can deduce the following formula for the conditional variance of $\tilde{V} - V$

$$E^W|\tilde{V} - V|^2 = \alpha_H E^B \Bigg(\int_0^s \int_0^s |r-v|^{2H_0-2} \prod_{i=1}^d |B^i_{s-r} - B^i_{s-v}|^{2H_i-2} drdv$$
$$+ \int_0^t \int_0^t |r-v|^{2H_0-2} \prod_{i=1}^d |B^i_{t-r} - B^i_{t-v}|^{2H_i-2} drdv$$
$$-2 \int_0^s \int_0^t |r-v|^{2H_0-2} \prod_{i=1}^d |B^i_{s-r} - B^i_{t-v} + x_i - y_i|^{2H_i-2} drdv \Bigg)$$
$$:= \alpha_H C(s,t,x,y). \tag{6.2}$$

Step 2 Fix $1 \le j \le d$. Let us estimate $C(s,t,x,y)$ when $s = t$, and $x_i = y_i$ for all $i \ne j$. We can write

$$C(t,t,x,y) = 2 \int_0^t \int_0^t |r-v|^{\kappa-1} \prod_{i \ne j}^d E\big(|\xi|^{2H_i-2}\big) E\big(|\xi|^{2H_j-2} - |z+\xi|^{2H_j-2}\big) drdv, \tag{6.3}$$

where $z = \dfrac{x_j - y_j}{\sqrt{|r-v|}}$ and ξ is a standard normal variable. Set $\beta_j = 2H_j + 1 > 2$. By Lemma 9.6 the factor $E\big(|\xi|^{2H_j-2} - |z+\xi|^{2H_j-2}\big)$ can be bounded by a constant if $|r-v| \le (x_j - y_j)^2$, and it can be bounded by $C|x_j - y_j|^{\beta_j} |r-v|^{-\beta_j/2}$ if $|r-s| > (x_j - y_j)^2$. In this way we obtain

$$C(t,t,x,y) \le C \int_{\{0<r,v<t, |r-v| \le (x_j-y_j)^2\}} |r-v|^{\kappa-1} drdv$$
$$+ C|x_j - y_j|^{\beta_j} \int_{\{0<r,v<t, |r-v| > (x_j-y_j)^2\}} |r-v|^{\kappa-1-\beta_j/2} drdv$$
$$\le C|x_j - y_j|^{2\kappa}.$$

So, from (6.1) we have

$$E^W |u(t,x) - u(t,y)|^p \le C|x_j - y_j|^{\kappa p}. \tag{6.4}$$

Step 3 Suppose now that $s < t$, and $x = y$. Set $\delta = \sum_{i=1}^d H_i - d$. We have

$$C(s,t,x,x)$$
$$= C \Bigg[\int_s^t \int_s^t |r-v|^{\kappa-1} drdv$$
$$+ \int_0^s \int_0^t |r-v|^{2H_0-2} \big(|r-v|^\delta - |r-v+t-s|^\delta\big) drdv \Bigg].$$

The first integral is $O((t-s)^{\kappa+1})$, when $t-s$ is small. For the second integral we use the change of variable $\sigma = r - v, v = \tau$, and we have

$$\begin{aligned}
&\int_0^s \int_0^t |r-v|^{2H_0-2}\big(|r-v|^\delta - |r-v+t-s|^\delta\big) dr dv \\
\leq &\int_0^t d\tau \int_{-t}^s |\sigma|^{2H_0-2}\big||\sigma|^\delta - |\sigma+t-s|^\delta\big| d\sigma \\
= &t\Big[\int_0^s \sigma^{2H_0-2}\big(\sigma^\delta - (\sigma+t-s)^\delta\big) d\sigma \\
&+ \int_{-t}^{s-t} (-\sigma)^{2H_0-2}\big((-\sigma-t+s)^\delta - (-\sigma)^\delta\big) d\sigma \\
&+ \int_{s-t}^0 (-\sigma)^{2H_0-2}\big|(-\sigma)^\delta - (\sigma+t-s)^\delta\big| d\sigma\Big] \\
= &t[A' + B' + C'].
\end{aligned}$$

For the first term in the above decomposition we can write

$$\begin{aligned}
A' &= (t-s)^{\kappa-1} \int_0^{\frac{t_1}{t-s}} \sigma^{2H_0-2}\big(\sigma^\delta - (\sigma+1)^\delta\big) d\sigma \\
&\leq (t-s)^{\kappa-1} \int_0^\infty \sigma^{2H_0-2}\big(\sigma^\delta - (\sigma+1)^\delta\big) d\sigma \\
&\leq C(t-s)^\kappa,
\end{aligned}$$

because $2H_0 + \sum_{i=1}^d -d - 3 < -1$. Similarly we can get that

$$B' \leq (t-s)^\kappa \int_1^\infty \sigma^{2H_0-2}\big(\sigma^\delta - (\sigma+1)^\delta\big) d\sigma.$$

At last,

$$C' \leq \int_0^{t-s} \sigma^{2H_0-2}\big(\sigma^\delta + (t-s-\sigma)^\delta\big) d\sigma = C(t-s)^\kappa.$$

So we have

$$E^W|u(s,x) - u(t,y)|^p \leq C(t-s)^{\frac{\kappa}{2}p}. \tag{6.5}$$

Step 4 Combining Equation 6.4 and Equation 6.5 with the estimates (6.1) and (6.2), the result of this theorem now can be concluded from Theorem 1.4.1 in Kunita [8] if we choose p large enough. ■

7 Case $H_0 > \frac{3}{4}, H_1 = \frac{1}{2}$ and $d = 1$

7.1 Preliminaries

In this case, all the setup is the same as before except that if ϕ and ψ are functions in $\mathcal{E}$, then

$$
\begin{aligned}
E\left(W(\phi)W(\psi)\right) &= \langle \phi, \psi\rangle_{\mathcal{H}} = \alpha_{H_0} \\
&\times \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \phi(s,x)\psi(t,x)|s-t|^{2H_0-2} dsdtdx,
\end{aligned}
$$

where $\alpha_{H_0} = H_0(2H_0-1)$.

7.2 Definition and exponential integrability of $\int_0^t \int_{\mathbb{R}} \delta(B^x_{t-r} - y) W(dr, dy)$

Similarly we have the following theorem as well.

Theorem 7.1 *Suppose that $H_1 = 1/2$ and $H_0 > 3/4$. Then, for any $\varepsilon > 0$ and $\delta > 0$, $A^{\varepsilon,\delta}_{t,x}$ defined in (3.2) belongs to $\mathcal{H}$ and the family of random variables $V^{\epsilon,\delta}_{t,x}$ defined in (3.3) converges in L^2 to a limit denoted by*

$$
V_{t,x} = \int_0^t \int_{\mathbb{R}} \delta(B^x_{t-r} - y) W(dr, dy)\,. \tag{7.6}
$$

Conditional to B, $V_{t,x}$ is a Gaussian random variable with mean 0 and variance

$$
\mathrm{Var}^W(V_{t,x}) = \alpha_{H_0} \int_0^t \int_0^t |r-s|^{2H_0-2} \delta\left(B_r - B_s\right) drds\,. \tag{7.7}
$$

Proof Fix ε, ε', δ and $\delta' > 0$.

$$
\begin{aligned}
&E^B E^W \left(V^{\epsilon,\delta}_{t,x}, V^{\epsilon',\delta'}_{t,x}\right) = E^B \left\langle A^{\varepsilon,\delta}_{t,x}, A^{\varepsilon',\delta'}_{t,x}\right\rangle_{\mathcal{H}} \\
&\quad = \alpha_{H_0} E^B \Bigg(\int_{[0,t]^4} \int_{\mathbb{R}} p_\epsilon(B^x_s - y) p_{\epsilon'}(B^x_r - y) \\
&\qquad \times \varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)|u-v|^{2H_0-2} dydudvddsdr \Bigg) \\
&\quad = \alpha_{H_0} \Bigg(\int_{[0,t]^4} E^B p_{\epsilon+\epsilon'}(B_s - B_r) \\
&\qquad \times \varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)|u-v|^{2H_0-2} dudvdsdr \Bigg) \\
&\quad = \alpha_{H_0} \Bigg(\int_{[0,t]^4} \frac{1}{\sqrt{2\pi}} \left(\epsilon + \epsilon' + |s-r|\right)^{-1/2} \\
&\qquad \times \varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)|u-v|^{2H_0-2} dudvdsdr \Bigg).
\end{aligned}
$$

By Lemma 9.3,

$$\int_{[0,t]^2} (\epsilon + \epsilon' + |s-r|)^{-1/2} \times \varphi_\delta(t-s-u)\varphi_{\delta'}(t-r-v)|u-v|^{2H_0-2} ddudv$$
$$\leq C|s-r|^{2H_0-5/2}.$$

Then by the dominated convergence theorem, $E^B E^W \left(V_{t,x}^{\epsilon,\delta}, V_{t,x}^{\epsilon',\delta'} \right)$ converges to

$$\frac{\alpha_{H_0}}{\sqrt{2\pi}} \int_{[0,t]^2} |s-r|^{2H_0-5/2} dsdr$$

as ε, ε', δ, and δ' tend to zero. This implies that $V_{t,x}^{\epsilon,\delta}$ converges in L^2 as ε and δ tend to zero to a limit denoted by $V_{t,x}$. As ϵ, δ go to 0, $E^W \left[\left(V_{t,x}^{\epsilon,\delta} \right)^2 \right]$ converges to right side of Equation (7.7) almost surely, and because of the above argument, the convergence is also in L^1, and this implies Equation (7.7). ■

7.3 Feynman-Kac formula

By Proposition 3.3 and Theorem 6.2 in [6], we have the following theorem.

Theorem 7.2 *Suppose that $H_1 = 1/2$ and $H_0 > 3/4$. Then, for any $\lambda \in \mathbb{R}$, we have*

$$E \exp \left(\lambda \int_0^t \int_{\mathbb{R}} \delta(B_{t-r}^x - y) W(dr, dy) \right) < \infty \,,$$

and for any measurable and bounded function f the process

$$u(t,x) = E^B \left(f(B_t^x) \exp \left(\int_0^t \int_{\mathbb{R}} \delta(B_{t-r}^x - y) W(dr, dy) \right) \right) \tag{7.8}$$

is a weak solution to Equation (1.2).

7.4 Hölder continuity

We also have the following theorem, whose proof is similar to that of Theorem 6.1.

Theorem 7.3 *Suppose $H_1 = 1/2, H_0 > 3/4$ and let $u(t,x)$ be the solution of Equation (1.2). Then $u(t,x)$ has a continuous modification such that for any $\rho \in (0, H_0 - 3/4)$ and any compact rectangle $I \subset \mathbb{R}_+ \times \mathbb{R}$ there exists a positive random variable K_I such that almost surely, for any $(t_1,x_1),(t_2,x_2) \in I$ we have*

$$|u(t_2,x_2) - u(t_1,x_1)| \leq K_I(|t_2-t_1|^\rho + |x_2-x_1|^{2\rho}).$$

Proof As in the proof of Theorem 6.1, we have

$$E^W |u(s,x) - u(t,y)|^p \leq C_p \left[E^B E^W |\tilde{V} - V|^2 \right]^{p/2} ,$$

where $V=\int_0^t\int_{\mathbb{R}}\delta(B^x_{t-r}-z)W(dr,dz)$ and $\tilde V=\int_0^s\int_{\mathbb{R}}\delta(B^y_{s-r}-z)W(dr,dz)$. If $s=t$, we can write

$$\begin{aligned}E^BE^W|\tilde V-V|^2 &= 2\int_0^t\int_0^t|r-v|^{2H_0-2}\\&\quad\times E\left[\delta(B_r-B_v)-\delta(B_r-B_v+x-y)\right]drdv\\&= \frac{2}{\sqrt{2\pi}}\int_0^t\int_0^t|r-v|^{2H_0-5/2}(1-e^{-\frac{(x-y)^2}{2|r-v|}})drdv\,.\end{aligned}$$

For any $2\rho<\gamma<2H_0-3/2$, we have $1-e^{-\frac{(x-y)^2}{2|r-v|}}\le\left(\frac{(x-y)^2}{2|r-v|}\right)^\gamma$. Thus $E^BE^W|\tilde V-V|^2\le C_\gamma|x-y|^{2\gamma}$. Consequently, we have

$$E^W|u(t,x)-u(t,y)|^p\le C|x-y|^{\gamma p}. \tag{7.9}$$

On the other hand, if $x=y$,

$$\begin{aligned}E^BE^W|\tilde V-V|^2=C\Bigg[&\int_s^t\int_s^t|r-v|^{2H_0-5/2}drdv\\&+\int_0^s\int_0^t|r-v|^{2H_0-2}\big(|r-v|^{-1/2}-|r-v+t-s|^{-\frac12}\big)drds\Bigg],\end{aligned}$$

and by a similar computation as step 3 before, we can get

$$E^W|u(s,x)-u(t,x)|^p\le C(t-s)^{(H_0-3/4)p}. \tag{7.10}$$

Combining (7.9) and (7.10) we prove the theorem. ■

8 Skorohod type equations and chaos expansion

In this section we consider the following heat equation on $\mathbb{R}^d$

$$\begin{cases}\dfrac{\partial u}{\partial t}=\frac12\Delta u+u\diamond\frac{\partial^{d+1}}{\partial t\partial x_1\cdots\partial x_d}W\\ u(0,x)=f(x)\,.\end{cases} \tag{8.1}$$

The difference between the above equation and Equation (1.2) is that here we use the Wick product $\diamond$ (see [7], for example). This equation is studied in [6] in the case $H_1=\cdots=H_d=1/2$. As in that paper, we can define the following notion of solution.

Definition 8.1 *An adapted random field $u=\{u(t,x),t\ge0,x\in\mathbb{R}^d\}$ such that $E(u^2(t,x))<\infty$ for all (t,x) is a solution to Equation (8.1) if for any $(t,x)\in[0,\infty)\times\mathbb{R}^d$, the process $\{p_{t-s}(x-y)u(s,y)\mathbf{1}_{[0,t]}(s),s\ge0,y\in\mathbb{R}^d\}$ is Skorohod integrable, and the following equation holds*

$$u(t,x)=p_tf(x)+\int_0^t\int_{\mathbb{R}^d}p_{t-s}(x-y)u(s,y)\delta W_{s,y}, \tag{8.2}$$

where $p_t(x)$ denotes the heat kernel and $p_tf(x)=\int_{\mathbb{R}^d}p_t(x-y)f(y)dy$.

As in the paper [6] the solution $u(t,x)$ to (8.1) admits the following Wiener chaos expansion

$$u(t,x)=\sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)), \tag{8.3}$$

where I_n denotes the multiple stochastic integral with respect to W and $f_n(\cdot,t,x)$ is a symmetric element in $\mathcal{H}^{\otimes n}$, defined explicitly as

$$\begin{aligned} f_n(s_1,y_1,\ldots,s_n,y_n,t,x) &= \frac{1}{n!} \\ \times p_{t-s_{\sigma(n)}}(x-y_{\sigma(n)})\cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}&(y_{\sigma(2)}-y_{\sigma(1)})p_{s_{\sigma(1)}}f(y_{\sigma(1)}). \end{aligned} \tag{8.4}$$

In the above equation σ denotes a permutation of $\{1,2,\ldots,n\}$ such that $0<s_{\sigma(1)}<\cdots<s_{\sigma(n)}<t$.

The following is the main result of this section.

Theorem 8.2 *Suppose that $2H_0+\sum_{i=1}^d H_i>d+1$ and that f is a bounded measurable function. Then the process*

$$\begin{aligned} u(t,x) &= E^B\Bigg[f(B_t^x)\exp\Bigg(\int_0^t\int_{\mathbb{R}^d}\delta(B_{t-r}^x-y)W(dr,dy) \\ &\quad -\frac{1}{2}\alpha_H\int_0^t\int_0^t|r-s|^{2H_0-2}\prod_{i=1}^d\left|B_r^i-B_s^i\right|^{2H_i-2}drds\Bigg)\Bigg] \end{aligned} \tag{8.5}$$

is a weak solution to Equation (1.2).

Proof From Theorem 3.2 we obtain that the expectation E^B in Equation (8.5) is well defined. Then, it suffices to show that the random variable $u(t,x)$ has the Wiener chaos expansion (8.3). This can be easily proved by expanding the exponential and then taken the expectation with respect to B.

Theorem 3,1 implies that that almost surely $\delta(B_{t-\cdot}^x-\cdot)$ is and element of $\mathcal{H}$ with a norm given by (3.4). As a consequence, almost surely with respect to the Brownian motion B, we have the following chaos expansion for the exponential factor in Equation (8.5)

$$\begin{aligned} &\exp\Bigg(\int_0^t\int_{\mathbb{R}^d}\delta(B_{t-r}^x-y)W(dr,dy) \\ &-\frac{1}{2}\alpha_H\int_0^t\int_0^t|r-s|^{2H_0-2}\prod_{i=1}^d\left|B_r^i-B_s^i\right|^{2H_i-2}drds\Bigg)=\sum_{n=0}^{\infty}I_n(g_n)\,, \end{aligned}$$

where g_n is the symmetric element in $\mathcal{H}^{\otimes n}$ given by

$$g_n(s_1,y_1,\ldots,s_n,y_n,t,x)=\frac{1}{n!}\;\delta(B_{t-s_1}^x-y_1)\cdots\delta(B_{t-s_n}^x-y_n)\,. \tag{8.6}$$

Thus the right hand side of (8.5) admits the following chaos expansion

$$u(t,x)=\sum_{n=0}^{\infty}\frac{1}{n!}I_n(h_n(\cdot,t,x))\,, \tag{8.7}$$

with

$$h_n(t,x)=E^B\left[f(B_t^x)\delta(B_{t-s_1}^x-y_1)\cdots\delta(B_{t-s_n}^x-y_n)\right]\,. \tag{8.8}$$

This can be regarded as a Feynman-Kac formula for the coefficients of chaos expansion of the solution of (8.1). To compute the above expectation we shall use the following

$$\begin{aligned} E^B\left[f(B_t^x)\delta(B_t^x-y)\middle|\mathcal{F}_s\right] &= \int_{\mathbb{R}^d}p_{t-s}(B_s^x-z)f(z)\delta(z-y)dz \\ &= p_{t-s}(B_s^x-y)f(y)\,. \end{aligned} \tag{8.9}$$

Assume that $0<s_{\sigma(1)}<\cdots<s_{\sigma(n)}<t$ for some permutation σ of $\{1,2,\cdots,n\}$. Then conditioning with respect to $\mathcal{F}_{t-s_{\sigma(1)}}$ and using the Markov property of the Brownian motion we have

$$\begin{aligned} h_n(t,x) &= E^B\big\{E^B\big[\delta(B_{t-s_{\sigma(n)}}^x-y_{\sigma(n)}) \\ &\qquad \times\cdots\delta(B_{t-s_{\sigma(1)}}^x-y_{\sigma(1)})f(B_t^x)\big|\mathcal{F}_{t-s_{\sigma(1)}}\big]\big\} \\ &= E^B\left[\delta(B_{t-s_{\sigma(n)}}^x-y_{\sigma(n)})\cdots\delta(B_{t-s_{\sigma(1)}}^x-y_{\sigma(1)})p_{s_{\sigma(1)}}f(B_{t-s_{\sigma(1)}}^x)\right]\,. \end{aligned}$$

Conditioning with respect to $\mathcal{F}_{t-s_{\sigma(2)}}$ and using (8.9), we have

$$\begin{aligned} h_n(t,x) &= E^B\big\{E^B\big[\delta(B_{t-s_{\sigma(n)}^x}-y_{\sigma(n)}) \\ &\qquad \times\delta(B_{t-s_{\sigma(1)}}^x-y_{\sigma(1)})p_{s_{\sigma(1)}}f(B_{t-s_{\sigma(1)}}^x)\big]\big|\mathcal{F}_{t-s_{\sigma(2)}}\big\} \\ &= E^B\Bigg\{\delta(B_{t-s_{\sigma(n)}^x}-y_{\sigma(n)})\cdots\delta(B_{t-s_{\sigma(2)}}^x-y_{\sigma(2)}) \\ &\qquad \times E^B\left[\delta(B_{t-s_{\sigma(1)}}^x-y_{\sigma(1)})p_{s_{\sigma(1)}}f(B_{t-s_{\sigma(1)}}^x)\middle|\mathcal{F}_{t-s_{\sigma(2)}}\right]\Bigg\} \\ &= E^B\Bigg[\delta(B_{t-s_{\sigma(n)}^x}-y_{\sigma(n)})\cdots\delta(B_{t-s_{\sigma(2)}}^x-y_{\sigma(2)}) \\ &\qquad \times p_{s_{\sigma(2)}-s_{\sigma(1)}}(B_{t-s_{\sigma(2)}}^x-y_{\sigma(1)})p_{s_{\sigma(1)}}f(y_{\sigma(1)})\Bigg]\,. \end{aligned}$$

Continuing this way we shall find out that

$$h_n(t,x)=p_{t-s_{\sigma(n)}}(x-y_{\sigma(n)})\cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(y_{\sigma(2)}-y_{\sigma(1)})p_{s_{\sigma(1)}}f(y_{\sigma(1)})$$

which is the same as (8.4). ■

Remark 8.3 *The method of this section can be applied to obtain a Feynman-Kac formula for the coefficients of the chaos expansion of the solution to Equation (1.2):*

$$u(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(h_n(\cdot,t,x))\,,$$

with

$$\begin{aligned} h_n(t,x) &= E^B\Bigg[f(B_t^x)\delta(B_{t-s_1}^x - y_1)\cdots\delta(B_{t-s_n}^x - y_n) \\ &\qquad \times \exp\left(\frac{1}{2}\alpha_H \int_0^t \int_0^t |r-s|^{2H_0-2} \prod_{i=1}^{d} \left|B_r^i - B_s^i\right|^{2H_i-2} drds \right) \Bigg]. \end{aligned} \tag{8.10}$$

Remark 8.4 *We can also consider Equatoin (1.2) when $d=1$, $H_1 = 1/2$ and $H_0 > 3/4$. In this case we see easily that the solution $u(t,x)$ admits the following chaos expansion*

$$u(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(h_n(\cdot,t,x))\,,$$

with

$$\begin{aligned} h_n(t,x) &= E^B\Bigg[f(B_t^x)\delta(B_{t-s_1}^x - y_1)\cdots\delta(B_{t-s_n}^x - y_n) \\ &\qquad \times \exp\left(\frac{1}{2}\alpha_{H_0} \int_0^t \int_0^t |r-s|^{2H_0-2} \delta\left(B_r - B_s\right) drds \right) \Bigg]. \end{aligned} \tag{8.11}$$

9 Appendix

Lemma 9.1 *Suppose $0 < \alpha < 1, \epsilon > 0, x > 0$, and that X is a standard normal random variable. Then there is a constant C independent of x and ϵ (it may depend on α) such that*

$$E|x + \epsilon X|^{-\alpha} \le C \min(\epsilon^{-\alpha}, x^{-\alpha})\,.$$

Proof It is straightforward to check that $K = \sup_{z\ge 0} E|z + X|^{-\alpha} < \infty$. Thus

$$E|x + \epsilon X|^{-\alpha} = \epsilon^{-\alpha} E|\frac{x}{\epsilon} + X|^{-\alpha} \le K\epsilon^{-\alpha}\,. \tag{9.12}$$

On the other hand,

$$
\begin{aligned}
E|x+\epsilon X|^{-\alpha} &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|x+\epsilon y|^{-\alpha}e^{-\frac{y^2}{2}}dy \\
&= \frac{1}{\sqrt{2\pi}}\left(\int_{\{|x+\epsilon y|>\frac{x}{2}\}}|x+\epsilon y|^{-\alpha}e^{-\frac{y^2}{2}}dy\right. \\
&\quad \left. + \int_{\{|x+\epsilon y|\le\frac{x}{2}\}}|x+\epsilon y|^{-\alpha}e^{-\frac{y^2}{2}}dy\right).
\end{aligned}
$$

It is easy to see that the first integral is bounded by $Cx^{-\alpha}$ for some constant C. The second integral, denoted by B is bounded as follows.

$$
\begin{aligned}
B &= C\frac{1}{\epsilon}\int_{|z|<\frac{x}{2}}|z|^{-\alpha}e^{-\frac{(z-x)^2}{2\epsilon^2}}dz \le C\frac{1}{\epsilon}\int_{|z|<\frac{x}{2}}|z|^{-\alpha}e^{-\frac{x^2}{8\epsilon^2}}dz \\
&= C\frac{x}{\epsilon}e^{-\frac{x^2}{8\epsilon^2}}x^{-\alpha} \le Cx^{-\alpha}.
\end{aligned}
$$

Thus we have $E|x+\epsilon X|^{-\alpha} \le C|x|^{-\alpha}$. Combining this with (9.12), we obtain the lemma. ■

Lemma 9.2 *Suppose $\alpha \in (0,1)$. There exists a constant $C>0$, such that*

$$
\sup_{\epsilon,\epsilon'}\int_{\mathbb{R}^2}p_\epsilon(x_1+y_1)p_{\epsilon'}(x_2+y_2)|y_1-y_2|^{-\alpha}dy_1dy_2 \le C|x_1-x_2|^{-\alpha}.
$$

Proof We can write

$$
\int_{\mathbb{R}^2}p_\epsilon(x_1+y_1)p_{\epsilon'}(x_2+y_2)|y_1-y_2|^{-\alpha}dy_1dy_2 = \mathbb{E}\left(|\varepsilon X_1 - x_1 - \varepsilon' X_2 + x_2|^{-\alpha}\right).
$$

Thus Lemma 9.2 follows directly from Lemma 9.1. ■

Lemma 9.3 *Suppose $\alpha \in (0,1)$. There exists a constant $C>0$, such that*

$$
\sup_{\delta,\delta'}\int_0^t\int_0^t \varphi_\delta(t-s_1-r_1)\varphi_{\delta'}(t-s_2-r_2)|r_1-r_2|^{-\alpha}dr_1dr_2 \le C|s_1-s_2|^{-\alpha}
$$

Proof Since

$$
p_\delta(x) \ge p_\delta(x)I_{[0,\sqrt{\delta}]}(x) = \frac{1}{\sqrt{2\pi\delta}}e^{-\frac{x^2}{2\delta}}I_{[0,\sqrt{\delta}]}(x) \ge \frac{1}{\sqrt{2\pi e}}\varphi_{\sqrt{\delta}}(x),
$$

the lemma follows from Lemma 9.2. ■

Lemma 9.4 *Suppose that $2H_0+\sum_{i=1}^d H_i > d+1$. Let $B^1,\dots,B^d$ be independent one-dimensional Brownian motions. Then we have*

$$
E\left(\int_0^t s^{2H_0-2}\prod_{i=1}^d |B_s^i|^{2H_i-2}ds\right)^2 < \infty.
$$

Proof We can write

$$E\left(\int_0^t s^{2H_0-2}\prod_{i=1}^d |B_s^i|^{2H_i-2}ds\right)^2 = 2\int_0^t\int_0^s (sr)^{2H_0-2}$$
$$\times \prod_{i=1}^d E(|B_s^i|^{2H_i-2}|B_r^i|^{2H_i-2})drds$$

Let X be a standard normal random variable. From Lemma 9.1, taking into account that $2-2H_i<1$, we have when $r<s$,

$$\begin{aligned} E(|B_r^i|^{2H_i-2}|B_s^i|^{2H_i-2}) &= E[|B_r^i|^{2H_i-2}E[|\sqrt{s-r}X+x|^{2H_i-2}|_{x=B_r^i}]] \\ &\leq CE[|B_r^i|^{2H_i-2}(s-r)^{H_i-1}) \\ &\leq Cr^{H_i-1}(s-r)^{H_i-1}. \end{aligned} \tag{9.13}$$

As a consequence, the conclusion of the lemma follows from the fact that

$$\int_0^t\int_0^s r^{2H_0+\sum_{i=1}^d H_i-d-2}s^{2H_0-2}(s-r)^{\sum_{i=1}^d H_i-d}drds<\infty,$$

because $2H_0+\sum_{i=1}^d H_i-d-2>-1$ and $\sum_{i=1}^d H_i-d>-1$. ■

Lemma 9.5 *Let $B^1,\dots,B^d$ be independent one-dimensional Brownian motions. If $\alpha_i\in(-1,0), i=1,\dots d$, and $\sum_{i=1}^d \alpha_i>-2$, then $E\exp\left(\lambda\int_0^1\prod_{i=1}^d |B_s^i|^{\alpha_i}ds\right)<\infty$ for all $\lambda>0$.*

Proof The proof is based on the method of moments. We can write

$$\begin{aligned} E\exp\left(\lambda\int_0^1\prod_{i=1}^d |B_s^i|^{\alpha_i}ds\right) &= \sum_{n=1}^\infty \frac{\lambda^n}{n!}E\int_{[0,1]^n}\prod_{k=1}^n\prod_{i=1}^d |B_{s_k}^i|^{\alpha_i}ds \\ &= \sum_{n=1}^\infty \lambda^n\int_{[0<s_1<\cdots<s_n<1]}\prod_{i=1}^d E\left(\prod_{k=1}^n |B_{s_k}^i|^{\alpha_i}\right)ds. \end{aligned}$$

From Lemma 9.1 since $\alpha_i\in(-1,0)$ we obtain

$$E\left[|B_{s_k}^i|^{\alpha_i}|\mathcal{F}_{s_{k-1}}^i\right]=E\left[|B_{s_k}^i-B_{s_{k-1}}^i+B_{s_{k-1}}^i|^{\alpha_i}|\mathcal{F}_{s_{k-1}}^i\right]\leq C(s_k-s_{k-1})^{\alpha_i/2},$$

where $\mathcal{F}_t$ is the filtration generated by the Brownian motion B^i. As a consequence, taking the conditional expectation of $\prod_{k=1}^n |B_{s_k}^i|^{\alpha_i}$ with respect to the σ-fields $\mathcal{F}_{s_{n-1}}^i,\mathcal{F}_{s_{n-2}}^i,\dots,\mathcal{F}_{s_1}^i$ and $\mathcal{F}_0^i$, we get

$$E\left(\prod_{k=1}^n |B_{s_k}^i|^{\alpha_i}\right)\leq C^n(s_n-s_{n-1})^{\alpha_i/2}\dots(s_2-s_1)^{\alpha_i/2}s_1^{\alpha_i/2}.$$

Let $\alpha = \sum_{i=1}^{d} \alpha_i$, then we have

$$E \exp\left(\lambda \int_0^1 \prod_{i=1}^{d} |B_s^i|^{\alpha_i} ds\right) \leq \sum_{n=1}^{\infty} (C\lambda)^n$$
$$\times \int_{[0<s_1<\cdots<s_n<1]} (s_n - s_{n-1})^{\alpha/2} \dots (s_2 - s_1)^{\alpha/2} s_1^{\alpha/2} ds.$$

Since $\alpha > -2$, the integrals on the right side are equal to $\dfrac{(\Gamma(\alpha/2+1))^n}{(n+n\alpha/2)\Gamma(n+n\alpha/2)}$, and the series converges for any $\lambda > 0$. ■

Lemma 9.6 *For any* $0 < \alpha < 1$ *define*

$$C_\alpha(y) = E\big(|\xi|^{-\alpha} - |y+\xi|^{-\alpha}\big),$$

where $y > 0$ *and* ξ *is a standard normal random variable. Then*

$$C_\alpha(y) \leq C \min\left(1, (y^2 + y^{3-\alpha})\right),$$

for some constant $C > 0$.

Proof Notice first that $C_\alpha(y) < C$ where $C > 0$ is a constant, since $\lim_{y\to\infty} E|y+\xi|^{-\alpha} = 0$. On the other hand, we can decompose the function $C_\alpha(y)$ as follows

$$\begin{aligned}
C_\alpha(y) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \big(|x|^{-\alpha} - |y+x|^{-\alpha}\big) e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \Bigg(\int_{\{x\geq 0\}\cup\{x\leq -y\}} \big(|x|^{-\alpha} - |y+x|^{-\alpha}\big) e^{-\frac{x^2}{2}} dx \\
&\quad + \int_{\{-y<x<0\}} \big(|x|^{-\alpha} - |y+x|^{-\alpha}\big) e^{-\frac{x^2}{2}} dx \Bigg) \\
&= \frac{1}{\sqrt{2\pi}} (A+B),
\end{aligned}$$

where A and B denote the first and second integrals, respectively, in the last second equation. For integral A we can write

$$\begin{aligned}
A &= \int_0^\infty \big(x^{-\alpha} - (x+y)^{-\alpha}\big)\big(e^{-x^2/2} - e^{-(x+y)^2/2}\big) dx \\
&\leq \int_0^\infty x^{-\alpha}(x+y)^{1-\alpha}[(x+y)^\alpha - x^\alpha] y e^{-\frac{x^2}{2}} dx.
\end{aligned}$$

Therefore,

$$A \leq \int_0^\infty x^{1-2\alpha}[(x+y)^\alpha - x^\alpha] y e^{-\frac{x^2}{2}} dx + \int_0^\infty x^{-\alpha}[(x+y)^\alpha - x^\alpha] y^{2-\alpha} e^{-\frac{x^2}{2}} dx.$$

For the first integral in the above expression we use the estimate $(x+y)^\alpha - x^\alpha \leq \alpha y x^{\alpha-1}$ and for the second we use $(x+y)^\alpha - x^\alpha \leq y^\alpha$. In this way we obtain

$$A \leq Cy^2,$$

for some constant $C > 0$. On the other hand,

$$B = \int_0^y x^{-\alpha}(e^{-\frac{x^2}{2}} - e^{-\frac{(x+y)^2}{2}})dx \leq \int_0^y x^{-\alpha}(x+y)ydx \leq Cy^{3-\alpha},$$

for some constant $C > 0$, which completes the proof of the lemma. ■

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